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Time-dependent multipole analysis

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Abstract. An exact time-dependent multipole analysis of macroscopic electromagnetic fields is given for sources which may not have Fourier transforms, eg, they may spread out indefinitely in the course of time or have nonzero limits in the remote past or future. The multipole fields are determined by charge and energy conservation and the homogeneous Maxwell's equations for vacuum fields, without the use of the inhomogeneous Maxwell's equations or constitutive equations for the source material. The moments at each time are integrals of sources over the region spacelike to the space origin at this time. Since neither spherical Bessel functions nor associated Legendre polynomials are used, only rational functions are needed in this analysis. The problem of relating vector sources and fields is simplified by transforming it to an exactly equivalent scalar one.

1. Introduction

Blatt and Weisskopf (1952), Jackson (1962) and others have given exact expressions for each frequency component of the multipole moments of distributions of charge, current and magnetization. Rose (1955) has presented a detailed derivation of some of these results. Here we derive from less restrictive assumptions exact expressions for the multipole moments at each time instead of each frequency. While Granzow (1966) considered some aspects of a time-dependent multipole analysis, he determined the fields from their boundary values on a sphere rather than from their sources.

This time-dependent multipole analysis includes the usual ones as special cases and has certain advantages:

(i) The sources need not be confined to a fixed sphere over all time but can spread out indefinitely as do most sets of interacting or even non-interacting charged particles.

(ii) No assumptions are made about the behaviour of the sources in the remote past or future. They need not approach zero asymptotically or possess Fourier time transforms. As a result, the multipole fields from sources which approach nonzero values asymptotically can be obtained as a special case.

(iii) The only physical assumptions used to determine the multipole fields are charge and energy conservation and the homogeneous Maxwell's equations for fields in vacuum. Since neither constitutive equations for the source material nor the inhomogeneous Maxwell's equations are used, the results are valid whatever inhomogeneity, anisotropy, nonlinearity, hysteresis or other properties are possessed by the sources provided only that all contributions to the macroscopic distributions of charge, current, polarization and magnetization are included.

(iv) The space–time region is located from which sources can contribute to the multipole moments at each time as well as the region in which fields are determined by these moments. This does for each multipole component of the fields what the retarded and advanced Green functions do for the sum of all the multipole components.

(v) The linear map from sources to each multipole component of the fields commutes with rotations in space, time displacements, inversion about the space origin and time reversal. To keep this symmetry explicit and use it most effectively, neither coordinates in three space nor particular basis functions over the surface of a sphere such as spherical harmonics are introduced.

(vi) Only rational numbers rather than other real or complex numbers are used in any essential way. (A factor of π in certain intermediate steps could be eliminated by minor rearrangement of the calculation.) This partly reflects the fact that all representations of the rotation group SO_3 are real (Schur and Frobenius 1906). Neither associated Legendre polynomials nor spherical Bessel functions are needed. In addition to computational convenience, this may have physical significance if electromagnetic theory were developed for a space–time whose local topology was not the usual manifold but instead allowed continuous coordinates from other number fields, which might be finite.

(vii) The problem of relating vector sources to vector fields is reduced to an exactly equivalent one of relating certain scalar sources to scalar waves.

Our basic physical assumptions are stated in § 2. In § 3, three sequences of linear operators on functions over space–time are defined and certain of their properties given. Section 4 states the principal results of this paper, including a construction for each multipole component of the moments and fields from a given source. These results are proved in § 5. To facilitate interpretation of the quantities and operators introduced, additional consequences of our assumptions are summarized in § 6 and compared with others.

2. Physical assumptions

Sources for electromagnetic fields consist of four macroscopic distributions over space–time: ρ, charge density in units of charge/volume; $\boldsymbol{j}$, current density in units of current/area; $\boldsymbol{P}$, polarization, or electric dipole density, in units of charge/area; $\boldsymbol{M}$, magnetization, or magnetic dipole density, in units of current/length. All sources vanish outside some sphere whose radius may change with time. Charge is conserved locally,

$$\nabla . \boldsymbol{j} + \rho' = 0, \tag{1}$$

where ∇ is the gradient operator and prime denotes time differentiation.

Electromagnetic fields consist of two three-vector fields: $\boldsymbol{B}$, magnetic field in units of force/current × length; $\boldsymbol{E}$, electric field in units of force/charge. The fields from sources (whether emitted or absorbed by them) are defined only outside a sphere enclosing the sources whose radius may change with time. These fields approach zero as the radius vector $\boldsymbol{r}$ increases without limit. They satisfy the homogeneous Maxwell's equations:

$$\nabla . \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} + \boldsymbol{B}' = \boldsymbol{0}, \tag{2a}$$

$$\nabla . \boldsymbol{E} = 0, \qquad \nabla \times \boldsymbol{B} - \boldsymbol{E}'/c^2 = \boldsymbol{0}, \tag{2b}$$

where c is the speed of light. In addition to the fields from sources, we also construct fields satisfying these homogeneous Maxwell's equations throughout all space–time without ever being emitted or absorbed. These are defined to be *entire* fields in analogy with entire functions over the complex plane. Unlike fields from sources, these entire fields need not approach zero with increasing r. Entire fields can be viewed as fields whose sources are at infinity.

Conservation of energy requires that the electromagnetic energy radiated from a source less the energy absorbed by it equals the work it does against electromagnetic forces. When the field is a sum of an entire field ($\boldsymbol{B}^\circ$, $\boldsymbol{E}^\circ$) and the field from the source ($\boldsymbol{B}, \boldsymbol{E}$), the work done by the source is also a sum of two parts, the work done against forces from the external entire field and the work done against forces from the field of the sources themselves. The energy flux density or Poynting vector is then the sum of three parts, one proportional to $\boldsymbol{E}^\circ \times \boldsymbol{B}^\circ$, one proportional to $\boldsymbol{E} \times \boldsymbol{B}$, and one proportional to the cross terms $\boldsymbol{E}^\circ \times \boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{B}^\circ$. The first integrates to zero since the incoming energy in any entire field equals the outgoing. The second integrates to the work done by the sources against forces from the fields they emit or absorb. The integral of the cross terms equals the work done by the sources against the forces due to the entire field:

$$-\int \mathrm{d}t\, \mathrm{d}^3r(\boldsymbol{j}\,.\,\boldsymbol{E}^\circ + \boldsymbol{P}'\,.\,\boldsymbol{E}^\circ + \boldsymbol{M}'\,.\,\boldsymbol{B}^\circ) = \int \mathrm{d}t\, \mathrm{d}^2\boldsymbol{r}\,.\,(\boldsymbol{E}^\circ \times \boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{B}^\circ)/\mu_0 \quad (3)$$

where $\mu_0 = 1/\epsilon_0 c^2$ is the permeability of the vacuum in units of force/current^2. (In SI units $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}; in electrostatic units $\mu_0 c^2 = 4\pi$; in electromagnetic units $\mu_0 = 4\pi$; to use gaussian units, set $\boldsymbol{B} = \boldsymbol{H}/c$ and $\mu_0 c^2 = 4\pi$.)

Equations (1), (2) and (3) complete our assumptions concerning electromagnetic fields and their sources. These are consistent with the additional assumption that fields everywhere satisfy

$$\nabla \times \boldsymbol{B} - \boldsymbol{E}'/c^2 = \mu_0(\boldsymbol{j} + \boldsymbol{P}' + \nabla \times \boldsymbol{M}) \qquad \text{and} \qquad \nabla\,.\,\boldsymbol{E}/c^2 = \mu_0(\rho - \nabla\,.\,\boldsymbol{P})$$

instead of equations (2*b*). Were this assumption made, equation (3) would follow from local energy conservation, $\nabla\,.\,\boldsymbol{S} + U' + W = 0$, with the Poynting vector

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} - \boldsymbol{M} \times \boldsymbol{B},$$

the field energy density $U = (B^2 + E^2/c^2)/2\mu_0 - \boldsymbol{M}\,.\,\boldsymbol{B}$, and the power density

$$W = \boldsymbol{j}\,.\,\boldsymbol{E} + \boldsymbol{P}'\,.\,\boldsymbol{E} + \boldsymbol{M}'\,.\,\boldsymbol{B}$$

at which rate energy is transferred from the field to its sources.

3. Mathematical preliminaries

Both the presentation of our results and their proofs are facilitated by introducing three sequences of linear integral and differential operators on real functions over space–time. We define $Xf|_{t,\boldsymbol{r}}$ to be the value at time t and position $\boldsymbol{r}$ of the function Xf obtained by operating with X on the function f.

For each non-negative integer L, we define the integral operator

$$G_L f|_{t,\boldsymbol{r}} = \frac{1}{4\pi r^{L+1}} \int \mathrm{d}^3 s P_L\left(\frac{\boldsymbol{r}\,.\,\boldsymbol{s}}{rs}\right) s^L f(t, \boldsymbol{s}) \qquad (4)$$

where P_L is the Legendre polynomial of degree L and the integral is over all space. This operator is one component in a multipole decomposition of the Green operator for Poisson's equation, ie, if $-\nabla^2 g = f$ and $f = 0$ outside some sphere, then $g = \Sigma_L G_L f$

outside this sphere. The domain of G_L includes all continuous functions which decrease faster than any negative power of r as r increases. When operating on any function in its domain, G_L produces a solution to Laplace's equation which is homogeneous of degree $-L-1$ in r. These properties can be stated algebraically by

$$\begin{aligned} \boldsymbol{r}\cdot\nabla G_L &= -(L+1)G_L, \qquad (\boldsymbol{r}\times\nabla)^2 G_L = G_L(\boldsymbol{r}\times\nabla)^2 = -L(L+1)G_L, \\ G_L\boldsymbol{r}\cdot\nabla &= -(L+3)G_L, \qquad \nabla^2 G_L = G_L\nabla^2 = 0, \end{aligned} \tag{5}$$

and if σ is any spherically symmetric function over space–time with $\int_0^\infty \mathrm{d}r r\sigma(t, r) = 1$ for all t, then

$$G_L\sigma G_K = 0 \qquad \text{for } L \neq K,$$

$$G_L\sigma G_L = \frac{G_L}{2L+1}.$$

For each non-negative integer L, we define another integral operator

$$I_L f|_{t,\boldsymbol{r}} = \frac{(2L+1)!!}{2^{L+1}L!}\int_{-1}^{1} \mathrm{d}u(1-u^2)^L f(t+ur/c, \boldsymbol{r}). \tag{6}$$

The value of $I_L f$ at time t and position $\boldsymbol{r}$ is a weighted average of values of f from $t-r/c$ to $t+r/c$ at position $\boldsymbol{r}$. The operator I_L is normalized so that it leaves unchanged functions which are constant over time, and the sequence of operators I_L converges to the unit operator as L increases without limit.

In addition to the integral operators G_L and I_L, we define a retardation operator K_L^+ and an advancement operator K_L^- which each combines a radius-dependent time displacement with L time differentiations. Denoting the operator of time differentiation by ∂, we define $K_L^\pm$ recursively by

$$K_0^\pm f|_{t\pm r/c,\boldsymbol{r}} = f|_{t,\boldsymbol{r}}, \tag{7a}$$

$$K_1^\pm = K_0^\pm(1 \pm r\partial/c), \tag{7b}$$

$$K_{L+1} = K_L + K_{L-1}\frac{r^2\partial^2}{(2L+1)(2L-1)c^2}. \tag{7c}$$

Equation (7*c*) is satisfied by both K_L^+ and K_L^- and whenever this is the case we leave the superscript implicit. The domain of K_L consists of all functions differentiable L times with respect to time. The operator K_L leaves unchanged functions which are constant over time. Both I_L and K_L are dimensionless but G_L has the dimensions of area.

The operators G_L, I_L and K_L all commute with the differential generators of rotations $\boldsymbol{r}\times\nabla$ and time displacements ∂, but not with $\boldsymbol{r}\cdot\nabla$, for it follows from equations (5), (6) and (7) that

$$[\boldsymbol{r}\cdot\nabla, G_L] = 2G_L, \tag{8a}$$

$$[\boldsymbol{r}\cdot\nabla, I_L] = (2L+1)(I_{L-1}-I_L) = I_{L+1}\frac{r^2\partial^2}{(2L+3)c^2}, \tag{8b}$$

$$[\boldsymbol{r}\cdot\nabla, K_L] = (2L+1)(K_L-K_{L+1}) = -K_{L-1}\frac{r^2\partial^2}{(2L-1)c^2}. \tag{8c}$$

The commutator $[\boldsymbol{r}\,.\,\nabla, K_L]$ can be evaluated by expressing K_L as

$$\exp\left(\mp\frac{r\partial}{c}\right)\frac{\theta_L(\pm r\partial/c)}{(2L-1)!!},$$

where θ_L is the polynomial with positive integral coefficients defined by Burchnall (1951). We then use $[\boldsymbol{r}\,.\,\nabla, f(r\partial/c)] = f'(r\partial/c)r\partial/c$, where f' is the derivative of f with respect to its argument, to obtain equation (8*c*). This equation remains valid for negative as well as positive L if the recursion relation (7) is used to extend the definition of K_L accordingly. The operator K_L for L negative is an integral instead of a differential operator and equation (8*b*) follows from $I_L = \frac{1}{2}(K^{+}_{-L-1} + K^{-}_{-L-1})$.

When operating on $\mathrm{e}^{-\mathrm{i}\omega t}$, I_L gives $(2L+1)!!j_L(kr)/(kr)^L$ and K_L gives

$$\mathrm{i}(kr)^{L+1}h_L(kr)/(2L-1)!!,$$

where $k = \omega/c$ and j_L and h_L are the spherical Bessel and Hankel functions defined by Morse and Feshbach (1953).

For large r, the asymptotic behaviour of $I_L f$ and $K_L f$ is

$$(\mp r\partial/c)^{L+1} I_L f|_{t\pm r/c.\boldsymbol{r}} \to \tfrac{1}{2}(2L+1)!!f|_{t.\boldsymbol{r}},$$
$$(2L-1)!!K_L^{\pm} f|_{t\pm r/c.\boldsymbol{r}} \to (\pm r\partial/c)^L f|_{t.\boldsymbol{r}}. \tag{9}$$

4. Electromagnetic fields from sources

Theorem. The outgoing fields emitted by sources (or the incoming fields absorbed by them) are uniquely determined by the assumptions of § 2. The following construction gives these fields:

Step 1. From the distributions ρ, $\boldsymbol{j}$, $\boldsymbol{P}$ and $\boldsymbol{M}$ construct two real functions over space–time,

$$\alpha = (\boldsymbol{r}\times\nabla)\,.\,(\boldsymbol{j}+\boldsymbol{P}'+\nabla\times\boldsymbol{M}), \tag{10a}$$

$$\beta = -(\boldsymbol{r}\,.\,\nabla+2)(\rho-\nabla\,.\,\boldsymbol{P})-\boldsymbol{r}\,.\,(\boldsymbol{j}+\boldsymbol{P}'+\nabla\times\boldsymbol{M})'/c^2. \tag{10b}$$

These two functions defined locally from $\rho-\nabla\,.\,\boldsymbol{P}$, $\boldsymbol{j}+\boldsymbol{P}'+\nabla\times\boldsymbol{M}$ and their first derivatives suffice to determine the electromagnetic fields outside any time-dependent sphere enclosing the sources. Were we to consider the fields inside such a sphere satisfying the inhomogeneous Maxwell's equations, then $\boldsymbol{r}\,.\,\boldsymbol{B}$ and $\boldsymbol{r}\,.\,\boldsymbol{E}$ would satisfy inhomogeneous scalar wave equations whose scalar sources would be proportional to the α and β of equation (10) respectively. For this reason, we call α the *magnetic source* and β the *electric source*.

Step 2. From the sources α and β of equation (10) and the integral operators G_L and I_L of equations (4) and (6), construct two real functions over space–time for each non-negative integer L by

$$p_L = G_L I_L \alpha,$$
$$q_L = G_L I_L \beta. \tag{11}$$

Since these functions satisfy Laplace's equation at each time and approach zero for increasing r, their values on any sphere determine them everywhere. Were we to expand these functions over a sphere as a sum of spherical harmonics Y_L^M, the expansion coefficients would be proportional to the usual multipole moments at each time. Rather than expand p_L and q_L as a sum of particular basis functions, we work with the functions themselves and call them the multipole components of the *magnetic moment* and *electric moment* of the source.

Step 3. From the moments p_L and q_L of equation (11) and the integral operators K_L of equation (7) construct incoming or outgoing waves,

$$\begin{aligned} \psi_L &= K_L p_L \\ \phi_L &= K_L q_L. \end{aligned} \tag{12}$$

These functions are solutions to the homogeneous wave equation and we call them the *magnetic* and *electric waves* respectively. They differ from the Hertz–Debye potentials by the factor $L(L+1)$ (Debye 1909). They are related to the radial components of the fields by $\mu_0 \Sigma \psi_L = \boldsymbol{r} \cdot \boldsymbol{B}$, $\mu_0 c^2 \Sigma \phi_L = \boldsymbol{r} \cdot \boldsymbol{E}$.

Step 4. From the incoming or outgoing waves ψ_L and ϕ_L, construct the electromagnetic fields

$$\begin{aligned} \boldsymbol{B}_L &= -\frac{[\nabla \times (\boldsymbol{r} \times \nabla)\psi_L + \boldsymbol{r} \times \nabla \phi'_L]\mu_0}{L(L+1)}, \\ \boldsymbol{E}_L &= \frac{[\boldsymbol{r} \times \nabla \psi'_L - c^2 \nabla \times (\boldsymbol{r} \times \nabla)\phi_L]\mu_0}{L(L+1)}, \end{aligned} \tag{13}$$

for $L \neq 0$ and for $L = 0$:

$$\boldsymbol{B}_0 = \boldsymbol{0}, \qquad \boldsymbol{E}_0 = \mu_0 c^2 \phi_0 \boldsymbol{r}/r^2 = -\mu_0 c^2 \nabla \phi_0.$$

Equations (12) and (13) define waves as an intermediate step in constructing fields from moments. While these waves are useful in our proofs and for certain interpretations of our results, the fields can also be expressed directly in terms of the moments with the radial parts of the fields and the contributions to the radiation fields separated. To do this, we use equation (8*c*) and the vector identity

$$r^2 \nabla \times (\boldsymbol{r} \times \nabla) = \hat{\boldsymbol{r}}(\boldsymbol{r} \times \nabla)^2 + \boldsymbol{r} \times (\boldsymbol{r} \times \nabla)(\boldsymbol{r} \cdot \nabla + 1)$$

to obtain

$$\begin{aligned} \boldsymbol{B}_L &= -\frac{\mu_0}{L(L+1)}\left(LK_L \nabla p_L + K_L \boldsymbol{r} \times \nabla q'_L - \frac{1}{(2L-1)c^2} K_{L-1} \boldsymbol{r} \times (\boldsymbol{r} \times \nabla) p''_L\right), \\ \boldsymbol{E}_L &= -\frac{\mu_0}{L(L+1)}\left(Lc^2 K_L \nabla q_L - K_L \boldsymbol{r} \times \nabla p'_L - \frac{1}{2L-1} K_{L-1} \boldsymbol{r} \times (\boldsymbol{r} \times \nabla) q''_L\right). \end{aligned} \tag{14}$$

Only the first terms on the right of equations (14) contribute to the radial parts of $\boldsymbol{B}$ and $\boldsymbol{E}$ and only the second and third terms contribute to the radiation fields at large r. The gradients of the moments and their first $L+1$ time derivatives at the space–time point $(t, \boldsymbol{r})$ determine the incoming field at $(t - r/c, \boldsymbol{r})$ and the outgoing field at $(t + r/c, \boldsymbol{r})$.

5. Proofs

To prove the theorem of § 4 and establish the validity of the construction given for the fields, we prove three lemmas. The first establishes that every solution to the homogeneous Maxwell's equations (2) can be derived from moments by equations (12) and (13). The second shows that certain entire fields satisfying the homogeneous Maxwell's equations throughout space–time can be derived from certain real functions. The third shows that energy conservation expressed by equation (3) is satisfied in the presence of these entire fields if and only if the moments are those given by equation (11).

Lemma 1. If

$$\begin{aligned} \nabla^2 p_L &= 0, \qquad \boldsymbol{r}\cdot\nabla p_L = -(L+1)p_L, \qquad p_0' = 0, \\ \nabla^2 q_L &= 0, \qquad \boldsymbol{r}\cdot\nabla q_L = -(L+1)q_L, \qquad q_0' = 0, \end{aligned} \tag{15}$$

then the fields obtained by equations (12) and (13) satisfy the homogeneous Maxwell's equations (2). All fields satisfying equations (2) which approach zero for increasing r are a sum over non-negative L of incoming and outgoing fields obtained from moments satisfying equations (15).

Proof. From the vector identity $r^2\nabla^2 = \boldsymbol{r}\cdot\nabla(\boldsymbol{r}\cdot\nabla+1)+(\boldsymbol{r}\times\nabla)^2$, and the commutators $[\boldsymbol{r}\cdot\nabla, r^2] = 2r^2$ and $[\boldsymbol{r}\cdot\nabla, K_L]$ from equation (8*c*), we obtain the commutator of K_L with the second order differential operator $\Box = \partial^2/c^2-\nabla^2$,

$$[\Box, K_L] = \left(-K_L+\frac{2}{2L-1}K_{L-1}(\boldsymbol{r}\cdot\nabla+L+1)\right)\frac{\partial^2}{c^2}.$$

Thus,

$$\Box K_L p_L = (K_L\Box+[\Box, K_L])p_L = -K_L\nabla^2 p_L+\frac{2}{2L-1}K_{L-1}(\boldsymbol{r}\cdot\nabla+L+1)\frac{p_L''}{c^2}.$$

By the hypothesis of the lemma, $\nabla^2 p_L = 0$ and $\boldsymbol{r}\cdot\nabla p_L = -(L+1)p_L$ so that $\Box K_L p_L = 0$ and similarly $\Box K_L q_L = 0$. Since $K_L p_L$ and $K_L q_L$ are solutions to the wave equation, the fields obtained from them by equations (13) satisfy Maxwell's equation (2).

To show that arbitrary fields $\boldsymbol{B}$and $\boldsymbol{E}$ satisfying Maxwell's equations (2) and approaching zero for increasing r can be obtained from moments, we form $\dot{\boldsymbol{r}}\cdot\boldsymbol{B}$ and $\boldsymbol{r}\cdot\boldsymbol{E}$. From Maxwell's equations (2) and the vector identity

$$\nabla^2\boldsymbol{r}\cdot\boldsymbol{u} = (\boldsymbol{r}\cdot\nabla+2)\nabla\cdot\boldsymbol{u}-\boldsymbol{r}\times\nabla\cdot\nabla\times\boldsymbol{u},$$

it follows that $\boldsymbol{r}\cdot\boldsymbol{B}$ and $\boldsymbol{r}\cdot\boldsymbol{E}$ satisfy the wave equations $\Box\boldsymbol{r}\cdot\boldsymbol{B} = 0$ and $\Box\boldsymbol{r}\cdot\boldsymbol{E} = 0$. Each can be decomposed into an incoming and outgoing wave and each of these is uniquely determined by its values on the surface of a sphere. For convenience, we consider explicitly only the outgoing part of $\boldsymbol{r}\cdot\boldsymbol{B}$ at $t = 0$ and show that it can be obtained by operating on a suitable p_L with K_L^+. Similar arguments apply to the incoming as well as outgoing parts of $\boldsymbol{r}\cdot\boldsymbol{B}$ and $\boldsymbol{r}\cdot\boldsymbol{E}$ at any time. The values of $\boldsymbol{r}\cdot\boldsymbol{B}$ at $t = 0$ depend on the values of p_L and its first L time derivatives at $t = -r/c$. Since any continuous function over the surface of a sphere can be extended to a harmonic function outside the sphere which approaches zero for increasing r (eg Hobson 1931) and these can be expressed as a sum of p_L satisfying (15), we can obtain the outgoing part of $\boldsymbol{r}\cdot\boldsymbol{B}$ from a magnetic moment.

For a concise statement and proof of the second lemma it is useful to define an adjoint $X^\dagger$ for each operator X by

$$\int \mathrm{d}t\,\mathrm{d}^3r\,f(X^\dagger g) = \int \mathrm{d}t\,\mathrm{d}^3r\,g(Xf) \tag{16}$$

for all real functions f and g on which the particular operators are defined. Among the immediate consequences of definition (16) are $(XY)^\dagger = Y^\dagger X^\dagger$, $X^{\dagger\dagger} = X$, $(\boldsymbol{r}\times\nabla)^\dagger = -\boldsymbol{r}\times\nabla$, $(\boldsymbol{r}\cdot\nabla)^\dagger = -\boldsymbol{r}\cdot\nabla-3$, $\partial^\dagger = -\partial$, $I_L^\dagger = I_L$ and $K_L^{+\dagger} = K_L^-$. The adjoint of G_L is the operator

$$G_L^\dagger f|_{t,r} = \frac{r^L}{4\pi}\int \mathrm{d}^3 s\, P_L\left(\frac{\boldsymbol{r}\cdot\boldsymbol{s}}{rs}\right) s^{-L-1} f(t,s)$$

whose properties can be obtained from equations (5) by defining $G_{-L-1} = G_L^\dagger$.

Lemma 2. For any differentiable function f over space–time on which $I_L G_L^\dagger$ is defined, $I_L G_L^\dagger f$ is a solution of the wave equation $\Box I_L G_L^\dagger f = 0$. An entire electromagnetic field (satisfying Maxwell's equations (2) over all space–time) can be obtained by using $I_L G_L^\dagger f$ for either the magnetic or electric wave in equation (13).

Proof. The commutator $[\Box, I_L]$ can be determined analogously to $[\Box, K_L]$ in the proof of lemma 1, and it is

$$[\Box, I_L] = -\left(I_L + \frac{2}{2L+3} I_{L+1}(\boldsymbol{r}\cdot\nabla - L)\right)\frac{\partial^2}{c^2}.$$

Thus

$$\Box I_L G_L^\dagger = (I_L\Box + [\Box, I_L])G_L^\dagger = -\left(I_L\nabla^2 + \frac{2}{2L+3} I_{L+1}(\boldsymbol{r}\cdot\nabla - L)\frac{\partial^2}{c^2}\right) G_L^\dagger.$$

Since

$$\nabla^2 G_L^\dagger = (G_L\nabla^2)^\dagger = 0^\dagger = 0$$

and

$$(\boldsymbol{r}\cdot\nabla - L)G_L^\dagger = [G_L(\boldsymbol{r}\cdot\nabla - L)^\dagger]^\dagger = [G_L(-\boldsymbol{r}\cdot\nabla - L - 3)]^\dagger = 0^\dagger = 0$$

by equations (5) and (16), it follows that $\Box I_L G_L^\dagger f = 0$ for any f on which the operators are defined, so that substituting $I_L G_L f$ for the magnetic or electric wave in equation (13) gives an entire field.

Among the functions on which $I_L G_L^\dagger$ is defined are real functions which can be differentiated any number of times with respect to space and time and which equal zero outside some bounded space–time region (real C^∞ functions with compact support). We call these test functions.

The waves obtained from test functions by $\phi = I_L G_L^\dagger f$ come in from infinity and return entirely in the region spacelike to points at which f is nonzero. For fixed t, $\phi(t, r)$ is proportional to r^{L-1} though $\phi(t \pm r/c, r)$ decreases as r^{-1} for sufficiently large r.

Lemma 3. If $\boldsymbol{B}_L$ and $\boldsymbol{E}_L$ are derived from moments p_L and q_L, then equation (3) for energy conservation is satisfied for every entire field $\boldsymbol{B}^\circ$ and $\boldsymbol{E}^\circ$ derived from a test function if and only if p_L and q_L are related to sources by equations (10) and (11).

Proof. We consider explicitly only the determination of the electric moments from sources, since the magnetic moments can be similarly obtained. If $I_L G_L^\dagger f$ is an electric wave, then by equation (13)

$$\boldsymbol{B}_L^\circ = -\frac{\mu_0}{L(L+1)}\boldsymbol{r}\times\nabla I_L G_L^\dagger f',$$

$$\boldsymbol{E}_L^\circ = -\frac{\mu_0 c^2}{L(L+1)}\nabla\times(\boldsymbol{r}\times\nabla)I_L G_L^\dagger f.$$

The work done by sources against forces on them from this entire field is given by the left-hand side of equation (3). Expressing the entire fields in terms of the test function f and using equation (1) for charge conservation together with previously stated results, we obtain

$$-\int \mathrm{d}t\,\mathrm{d}^3r(\boldsymbol{j}\,.\,\boldsymbol{E}^\circ+\boldsymbol{P}'\,.\,\boldsymbol{E}^\circ+\boldsymbol{M}'\,.\,\boldsymbol{B}^\circ) = \frac{\mu_0}{L(L+1)}\int \mathrm{d}t\,\mathrm{d}^3r f' G_L I_L \beta. \tag{17}$$

The contribution to the radiated energy from the cross terms between the entire field and the field from the sources is given by the right-hand side of equation (3). While the integral is independent of which surface enclosing the sources is chosen, it is simplest to evaluate at large radii using the asymptotic expressions (9) for K_L and I_L. We express $\boldsymbol{B}$ and $\boldsymbol{E}$ in terms of the electric moment q_L by equations (12) and (13) and use the integral over solid angle

$$\int \mathrm{d}\Omega\,\boldsymbol{r}\,.\,[\nabla\times(\boldsymbol{r}\times\nabla)u]\times(\boldsymbol{r}\times\nabla v) = -L(L+1)\int \mathrm{d}\Omega\, v(\boldsymbol{r}\,.\,\nabla+1)u$$

which holds when at least one of the functions u or v is an eigenfunction of $(\boldsymbol{r}\times\nabla)^2$ with eigenvalue $-L(L+1)$, to obtain

$$\int \mathrm{d}t\,\mathrm{d}^2\boldsymbol{r}\,.\,(\boldsymbol{E}^\circ\times\boldsymbol{B}+\boldsymbol{E}\times\boldsymbol{B}^\circ) = \frac{\mu_0}{L(L+1)}\int \mathrm{d}t\,\mathrm{d}^3r f' q_L. \tag{18}$$

By equation (3) for energy conservation, equations (17) and (18) must be equal for all test functions f, hence $q_L = G_L I_L \beta$. Similarly $p_L = G_L I_L \alpha$, completing the proof of the lemma.

These three lemmas together constitute a proof of the theorem of § 4, for if every field satisfying Maxwell's equations (2) can be obtained from some moment and if just one set of moments is consistent with equations (1) and (3) for charge and energy conservation, then these three assumptions determine the field.

6. Interpretation and comparison

Except for normalization, the time-dependent moments constructed here are related to the moments $M_L^M(\omega)$ and $Q_L^M(\omega)$ defined by Blatt and Weisskopf (1952) and others by

$$p_L(t,\boldsymbol{r}) = \frac{1}{2\pi}\sum\int \mathrm{d}\omega\, M_L^M(\omega)Y_L^M(\theta,\phi)r^{-L-1}\,\mathrm{e}^{-\mathrm{i}\omega t},$$

$$q_L(t,\boldsymbol{r}) = \frac{1}{2\pi}\sum\int \mathrm{d}\omega\, Q_L^M(\omega)Y_L^M(\theta,\phi)r^{-L-1}\,\mathrm{e}^{-\mathrm{i}\omega t},$$

where the sum is over M from $-L$ to L and the integral is over all real ω.

In contrast with the moments at one frequency, the moments at one time necessarily depend on ρ and cannot be expressed in terms of $\nabla \times \boldsymbol{j}$ alone. French and Shimamoto (1953) considered an arbitrariness in the expressions for each frequency component of the moments. It is closely related to an arbitrariness in our definition of the scalar sources α and β by equation (10). Without changing either the moments or the fields obtained from these sources we can add to them any function of the form $\Box f$ on which $G_L I_L$ is defined, since by lemma 2, $G_L I_L \Box f = (\Box I_L G_L^{\dagger})^{\dagger} f = 0^{\dagger} f = 0$. For example, by combining equation (10*a*) with the vector identity

$$\boldsymbol{r} \times \nabla . \nabla \times \boldsymbol{M} = (\boldsymbol{r} . \nabla + 2)\nabla . \boldsymbol{M} - \boldsymbol{r} . \boldsymbol{M}''/c^2 + \Box \boldsymbol{r} . \boldsymbol{M},$$

we obtain an alternate equation for the magnetic source,

$$\alpha = \boldsymbol{r} \times \nabla . (\boldsymbol{j} + \boldsymbol{P}') + (\boldsymbol{r} . \nabla + 2)\nabla . \boldsymbol{M} - \boldsymbol{r} . \boldsymbol{M}''/c^2,$$

in which $\boldsymbol{M}$ enters in the same way $\boldsymbol{P}$ does in equation (10*b*) for the electric source β.

For sources in a sphere of radius R emitting or absorbing only one wavelength λ of radiation, terms in the multipole expansion with $L \gg R/\lambda$ contribute little to the sum. There is in general no single wavelength for the radiation so that we need a more general convergence criterion for this time-dependent analysis. To obtain one, we first define a norm for the space of all possible moments by the conditions:

(i) $|q_L| = |q|a^L$ for a point charge q fixed at position $\boldsymbol{a}$;

(ii) $|q_L|$ is invariant under rotations;

(iii) $|q_L|^2$ depends quadratically on q_L, that is,

$$|\mu p_L + \nu q_L|^2 + |\mu p_L - \nu q_L|^2 = 2\mu^2 |p_L|^2 + 2\nu^2 |q_L|^2$$

for all real numbers μ and ν and moments p_L and q_L. It then follows from equation (14) that the power integrated over all directions radiated at each time by a source is given by

$$\sum \frac{L+1}{4\pi L(2L+1)!!(2L-1)!!} \frac{\mu_0}{c^{2L}} \left(\frac{1}{c} |p_L^{(L+1)}|^2 + c|q_L^{(L+1)}|^2 \right),$$

where $p_L^{(L+1)}$ is the $(L+1)$st time derivative of p_L.

The units of $|p_L|$ are current × length^{L+1}, and the units of $|q_L|$ are charge × length^L. For each L, the ratios $|p_L^{(L+1)}|/|p_{L-1}^{(L)}|c$ and $|q_L^{(L+1)}|/|q_{L-1}^{(L)}|c$ are dimensionless. If N is an upper bound on these ratios for sufficiently large L, then the contribution to the radiated power from large L terms is bounded by a sum proportional to

$$\sum (L+1)N^{2L}/L(2L+1)!!(2L-1)!!.$$

For large L, this is approximately $\Sigma N^{2L}/(2L)!$ which converges for all N to approximately $e^N/2$. Terms with $L \gg N$ make negligible contribution to the sum, so this provides a more general convergence criterion. It reduces to $L \gg R/\lambda$ for radiation of wavelength λ from sources within a sphere of radius R since in this case, $N \simeq R/\lambda$.

When the source consists only of polarization and magnetization near the space origin, then

$$p_1(t, \boldsymbol{r}) = \frac{1}{2\pi r^3} \boldsymbol{r} . \int \mathrm{d}^3 s\, \boldsymbol{M}(t, \boldsymbol{s}) = \frac{1}{2\pi r^3} \boldsymbol{r} . \boldsymbol{m}(t),$$

$$q_1(t, \boldsymbol{r}) = \frac{1}{2\pi r^3} \boldsymbol{r} . \int \mathrm{d}^3 s\, \boldsymbol{P}(t, \boldsymbol{s}) = \frac{1}{2\pi r^3} \boldsymbol{r} . \boldsymbol{n}(t),$$

where $\boldsymbol{m}(t)$ and $\boldsymbol{n}(t)$ are the magnetic and electric dipole vectors at time t. It follows from our definitions that $|p_1| = |m|$ and $|q_1| = |n|$ and that the fields given by equation (14) with these moments are the Hertz dipole fields (Hertz 1889)

$$\boldsymbol{B}\left(t+\frac{r}{c}, \boldsymbol{r}\right) = \frac{\mu_0}{4\pi r^5}\left(3\boldsymbol{rr}\cdot\boldsymbol{m} - r^2\boldsymbol{m} + \frac{r}{c}(3\boldsymbol{rr}\cdot\boldsymbol{m}' - r^2\boldsymbol{m}') - r^2\boldsymbol{r}\times\boldsymbol{n}' + \frac{r^2}{c^2}\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{m}'') - \frac{r^3}{c}\boldsymbol{r}\times\boldsymbol{n}''\right),$$

$$\boldsymbol{E}\left(t+\frac{r}{c}, \boldsymbol{r}\right) = \frac{\mu_0}{4\pi r^5}\left(c^2(3\boldsymbol{rr}\cdot\boldsymbol{n} - r^2\boldsymbol{n}) + rc(3\boldsymbol{rr}\cdot\boldsymbol{n}' - r^2\boldsymbol{n}') + r^2\boldsymbol{r}\times\boldsymbol{m}' + r^2\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{n}'') + \frac{r^3}{c}\boldsymbol{r}\times\boldsymbol{m}''\right),$$

where $\boldsymbol{m}, \boldsymbol{n}$ and their time derivatives are to be evaluated at time t.

We obtain an adiabatic approximation to the moments and fields from sources with negligible radiation by keeping only those terms of lowest order in $1/c^2$ in equations (10) through (14). In this approximation, the operators I_L and K_L are unit operators so that the magnetic and electric moments are

$$p_L = G_L\boldsymbol{r}\times\boldsymbol{\nabla}\cdot(\boldsymbol{j}+\boldsymbol{P}'+\boldsymbol{\nabla}\times\boldsymbol{M}),$$

$$q_L = (L+1)G_L(\rho-\boldsymbol{\nabla}\cdot\boldsymbol{P}).$$

The electromagnetic fields in this approximation are

$$\boldsymbol{B}_L = -\frac{\mu_0}{L(L+1)}(L\boldsymbol{\nabla}p_L+\boldsymbol{r}\times\boldsymbol{\nabla}q_L'),$$

$$\boldsymbol{E}_L = -\frac{\mu_0 c^2}{L+1}\boldsymbol{\nabla}q_L = -\mu_0 c^2\boldsymbol{\nabla}G_L(\rho-\boldsymbol{\nabla}\cdot\boldsymbol{P}).$$

We conclude this section with the observation that the product of linear operators $K_L^{\pm}G_LI_L$ is one multipole component of the retarded or advanced Green operator for the scalar wave equation, that is, if $\Box g^{\pm} = f$, g^+ is an outgoing wave and g^- an incoming one, and $f = 0$ outside some sphere, then $g^{\pm} = \Sigma_L K_L^{\pm}G_LI_Lf$ outside this sphere. From the definitions of I_L, G_L and K_L, it follows that the sources which contribute to emission of each multipole component of the field to time t and position $\boldsymbol{r}$ are those in the region spacelike to the three-space origin at time $t-r/c$. Similarly, sinks which contribute to the absorption of each multipole component of the field from time t and position $\boldsymbol{r}$ are those in the region spacelike to the space origin at time $t+r/c$.

7. Conclusions

The multipole components of every electromagnetic field in vacuum can be obtained by equation (13) from corresponding components of two scalar solutions to the homogeneous wave equation, $\psi = \boldsymbol{r}\cdot\boldsymbol{B}/\mu_0$ and $\phi = \boldsymbol{r}\cdot\boldsymbol{E}/\mu_0c^2$. Equation (12) gives the outgoing parts of these waves at $t+r/c$ and the incoming parts at $t-r/c$ in terms of the magnetic and electric multipole moments at time t. The moments at each time are integrals of scalar sources over the region spacelike to the three-space origin at this time (equation (11)). The scalar sources at each space–time point depend on $\rho-\boldsymbol{\nabla}\cdot\boldsymbol{P}$, $\boldsymbol{j}+\boldsymbol{P}'+\boldsymbol{\nabla}\times\boldsymbol{M}$ and their first derivatives at this same point as specified in equation (10).

This method of multipole analysis is being applied to the study of classical scattering of electromagnetic radiation by moving dielectric spheres. The motions of the scatterers are less restricted than they would be for other methods of multipole analysis. The physical interpretation of the analysis is facilitated by relating each multipole component of the scattered radiation to the positions of the scatterers and their time derivatives at an earlier time, rather than relating the Fourier time transforms of each. Computational simplifications result from the transformation of the vector fields and sources to exactly equivalent scalar ones and the absence of Bessel functions and associated Legendre polynomials.

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